



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 21 Effective Duration

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SHIFTS IN YIELD CURVE & EFFECTIVE & KEY RATE DURATION



RECALL: MACAULAY, MODIFIED DURATION & CONVEXITY

- Duration measures a bond's price sensitivity to interest rate changes, assuming that the bond's cash flows do not change as interest rates change.



- **Duration is the linear approximation that considers only the first order term in the Taylor series.**
- **Convexity can be used to improve price changes estimated from modified duration.**
- **By considering convexity, we are accounting for the curvature of the yield-price curve through the second order term in the Taylor series.**

$$\left. \frac{dP}{P} \right|_{y_0} = \frac{P(y_0 + dy) - P(y_0)}{P(y_0)} = \frac{P'(y_0)}{P(y_0)} dy + \frac{1}{2} \frac{P''(y_0)}{P(y_0)} (dy)^2 + \dots$$

$$= -D_{\text{Mac}} \frac{dy}{(1 + y_0)} + C \left(\frac{dy}{1 + y_0} \right)^2. \text{ Setting } D_{\text{mod}} = \frac{D_{\text{Mac}}}{(1 + y_0)}; C_{\text{mod}} = \frac{2C}{(1 + y_0)^2}$$

$$= -D_{\text{mod}} dy + \frac{1}{2} C_{\text{mod}} (dy)^2 \text{ where } D_{\text{mod}} = \frac{P'(y_0)}{P(y_0)}; C_{\text{mod}} = \frac{P''(y_0)}{P(y_0)}$$

YIELD CURVE RISK & DURATION

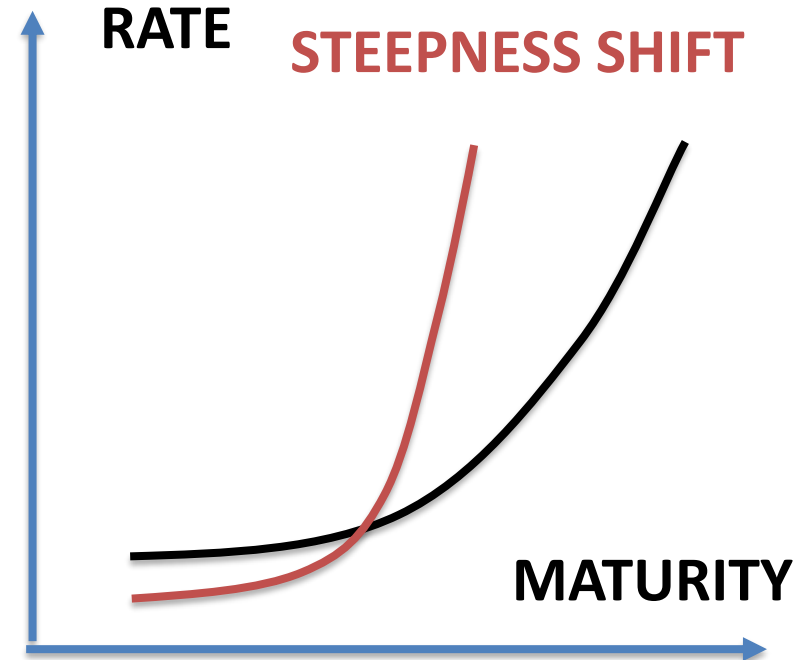
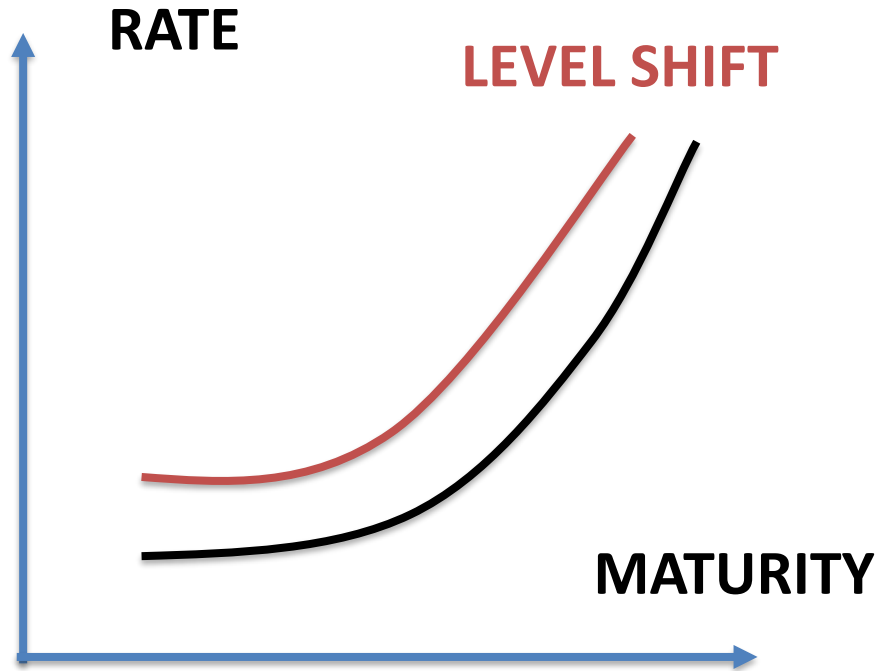
- Yield curve risk refers to risk arising from change in value of a bond portfolio due to unanticipated changes in the yield curve.



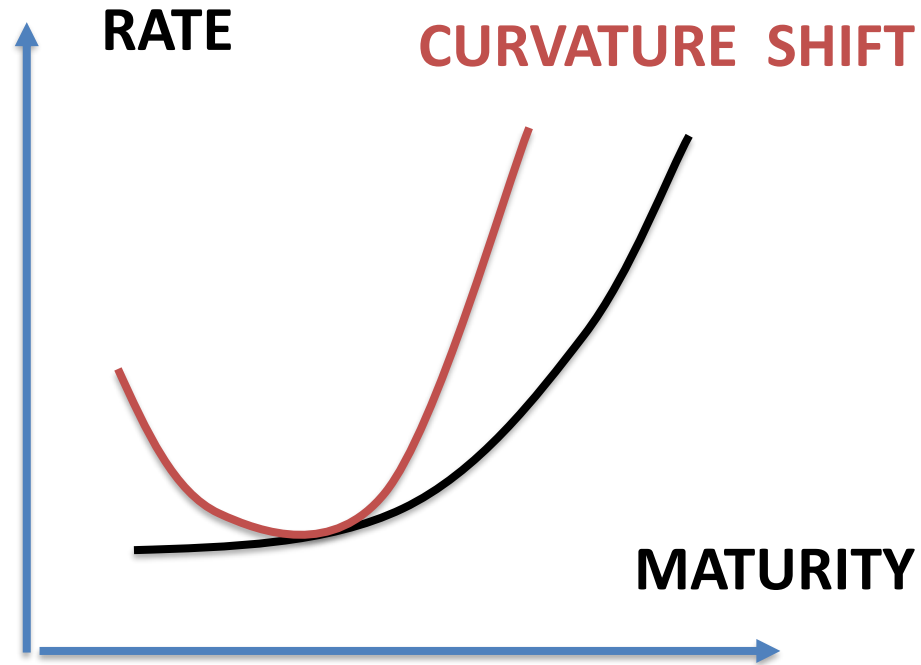
DECOMPOSITION OF YIELD CURVE SHIFTS

- Yield curve shifts can be decomposed into the following:
- **Level** (Δx_L) – A parallel increase or decrease of interest rates.
- **Steepness** (Δx_S) – Long-term interest rates increase while short-term rates decrease.
- **Curvature** (Δx_C) – Increasing curvature means short- and long-term interest rates increase while intermediate rates do not change.
- It has been empirically found that all yield curve movements can be described using a combination of one or more of these movements.

SHIFTING OF YIELD CURVE



CURVATURE OF YIELD CURVE



- We can then model the change in the value of our portfolio as follows:
- $$\frac{\Delta P}{P} = -D_L \Delta x_L - D_S \Delta x_S - D_C \Delta x_C$$
- where D_L , D_S and D_C are respectively the portfolio's sensitivities to changes in the yield curve's level, steepness, and curvature.
- For example, D_L is the percentage change in price of a bond corresponding a unit parallel shift in the yield curve.

SHAPING RISK

- Shaping risk refers to changes in portfolio value due to changes in the *shape* of the benchmark yield curve e.g. steepness & curvature.
- It covers the non-parallel shifts in the yield curve.



YIELD CURVE SENSITIVITY

- Yield curve sensitivity can be generally measured by
- effective duration, or
- key rate duration, or
- three-factor model that decomposes changes in the yield curve into changes in level, steepness, and curvature
- $$\frac{\Delta P}{P} = -D_L \Delta x_L - D_S \Delta x_S - D_C \Delta x_C$$

EFFECTIVE DURATION

- This is analogous to modified duration.
- Effective duration measures price sensitivity to small *parallel* shifts in the yield curve.
- It also assumes a flat yield curve to start with.
- It is important to note that effective duration is not an accurate measure of interest rate sensitivity to *non-parallel* shifts in the yield curve like those described by shaping risk.

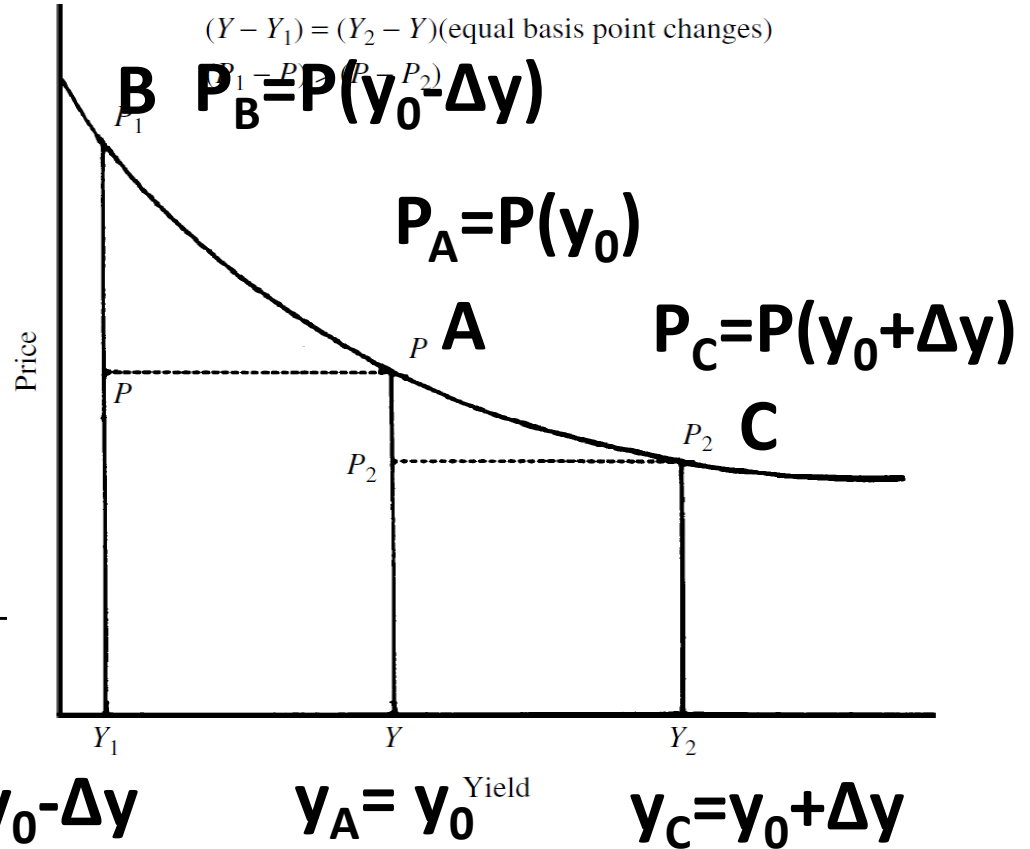


EFFECTIVE/MODIFIED DURATION $D_{\text{eff}}/D_{\text{mod}}$

$$P'(y_0) = \frac{P(y_0 + \Delta y) - P(y_0 - \Delta y)}{\Delta y - (-\Delta y)}$$

$$= \frac{P(y_0 + \Delta y) - P(y_0 - \Delta y)}{2\Delta y} = \frac{P_C - P_B}{(y_C - y_B)}$$

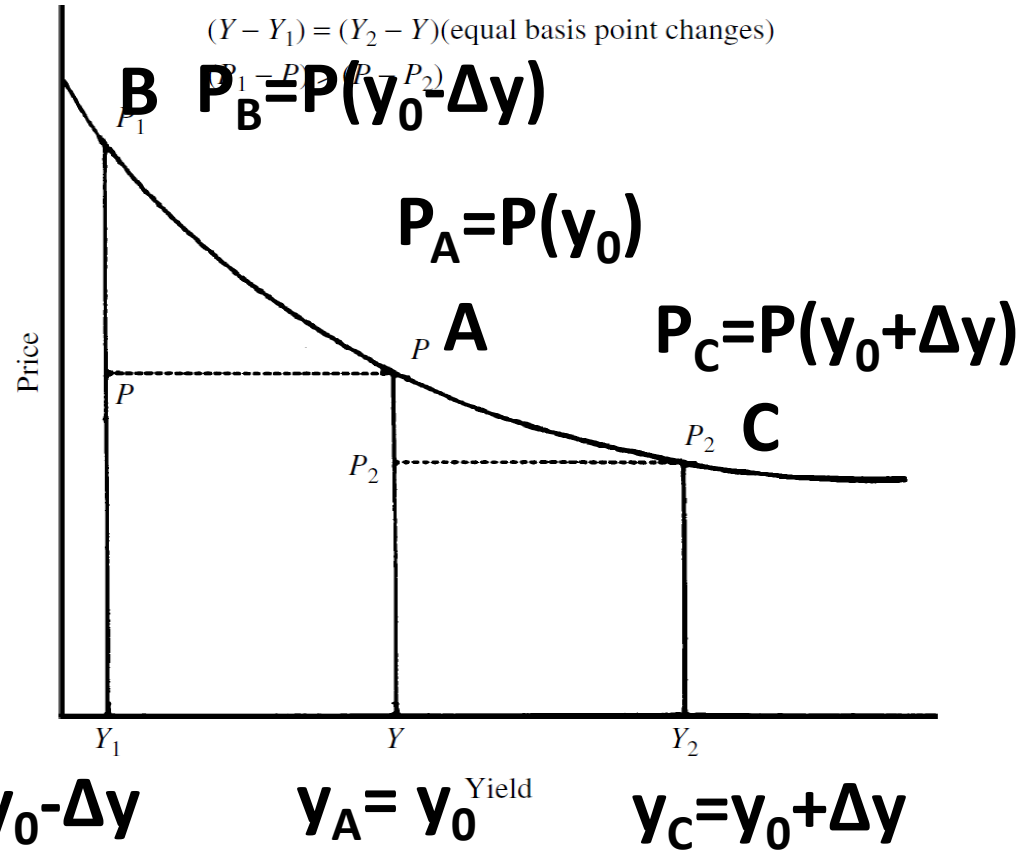
$$D_{\text{Mod}} = (-) \frac{P'(y_0)}{P(y_0)} = (-) \frac{P_C - P_B}{(y_C - y_B) \times P_A}$$



$P(y_0)$ is the current price of the bond.

$P(y_0 - \Delta y)$ is the price of the bond if YTM is *decreased* by Δy and

$P(y_0 + \Delta y)$ is the price of the bond if the YTM is *increased* by Δy .



- Note that $P(y_0 - \Delta y) > P(y_0 + \Delta y)$ because yield and price are inversely related.
- Because of the convexity of the price-yield relationship, the price increase $P(y_0 - \Delta y)$, for a given decrease in yield, is larger than the price decrease $P(y_0 + \Delta y)$.
- Then,
$$D_{\text{Mod}} = (-) \frac{P'(y_0)}{P(y_0)} = (-) \frac{P(y_0 + \Delta y) - P(y_0 - \Delta y)}{2\Delta y \times P(y_0)}$$

$$D_{\text{Mod}} = (-) \frac{P'(y_0)}{P(y_0)} = (-) \frac{P(y_0 + \Delta y) - P(y_0 - \Delta y)}{2\Delta y \times P(y_0)}$$

- The formula uses the average of the magnitudes of the price increase and the price decrease, which is why $P(y_0 - \Delta y) - P(y_0 + \Delta y)$ (in the numerator) is divided by 2 (in the denominator).
- $P(y_0)$ is in the denominator to convert this average price change to a percentage, and
- the Δy term is in the denominator to scale the duration measure to a 1% change in yield by convention.

$$D_{\text{Mod}} = (-) \frac{P'(y_0)}{P(y_0)} = (-) \frac{P(y_0 + \Delta y) - P(y_0 - \Delta y)}{2\Delta y \times P(y_0)}$$

- **Note that the Δy term in the denominator must be entered as a decimal (rather than in a whole percentage) to properly scale the duration estimate.**

EFFECTIVE/ MODIFIED CONVEXITY

**NOTE
REGARDING
PREFACTOR
OF 1/2**

$$\begin{aligned}P''(y_0) &= \frac{P'(y_0) - P'(y_0 - \Delta y)}{\Delta y} \\&= \frac{1}{\Delta y} \left[\frac{P(y_0 + \Delta y) - P(y_0)}{\Delta y} - \frac{P(y_0) - P(y_0 - \Delta y)}{\Delta y} \right] \\&= \frac{P(y_0 + \Delta y) + P(y_0 - \Delta y) - 2P(y_0)}{(\Delta y)^2} = \frac{P_C + P_B - 2P_A}{(\Delta y)^2}\end{aligned}$$

$$C_{\text{mod}} = \frac{P''(y_0)}{P(y_0)} = \frac{P_C + P_B - 2P_A}{(\Delta y)^2 P_A}$$

$$\left. \frac{dP}{P} \right|_{y_0} = \frac{P(y_0 + dy) - P(y_0)}{P(y_0)} = \frac{P'(y_0)}{P(y_0)} dy + \frac{1}{2} \frac{P''(y_0)}{P(y_0)} (dy)^2 + \dots$$

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$$= -D_{\text{mod}} dy + \frac{1}{2} C_{\text{mod}} (dy)^2 \text{ where } D_{\text{mod}} = \frac{P'(y_0)}{P(y_0)}; C_{\text{mod}} = \frac{P''(y_0)}{P(y_0)}$$

CALCULATION OF EFFECTIVE DURATION FOR BONDS WITH EMBEDDED OPTIONS

- ***Step 1:*** Given assumptions about benchmark interest rates, interest rate volatility, and any calls and/or puts, calculate the OAS for the bond using the current market price and the binomial model.
- ***Step 2:*** Impose a small parallel shift in the benchmark yield curve by an amount equal to $+\Delta y$.



- ***Step 3:*** Build a new binomial interest rate tree using the new yield curve.
- ***Step 4:*** Add the OAS from step 1 to each of the one-year rates in the interest rate tree to get a “modified” tree.
- ***Step 5:*** Compute $V_{y+\Delta y}$ using this modified interest rate tree.
- ***Step 6:*** Repeat steps 2 through 5 using a parallel rate shift of $-\Delta y$ to obtain a value of $V_{y-\Delta y}$.



COMPARISON OF EFFECTIVE DURATIONS OF CALLABLE, PUTTABLE, AND STRAIGHT BONDS.

- Both call and put options have the potential to reduce the life of a bond, so the duration of callable and puttable bonds will be less than or equal to the duration of their straight counterparts.
- While effective duration of straight bonds is relatively unaffected by small changes in interest rates, a decrease in rates would decrease the effective duration of a callable bond and increase that of a puttable bond.



- **Effective duration (callable) $\leq$ effective duration (straight).**
- **Effective duration (puttable) $\leq$ effective duration (straight).**
- **Effective duration (zero-coupon) $\approx$ maturity of the bond.**
- **Effective duration of fixed-rate coupon bond $<$ maturity of the bond.**
- **Effective duration of floater $\approx$ time (in years) to next reset.**

KEY RATE DURATION

